AXIAL CONTRACTION IN TOROIDAL PINCHES
WITH NON-CIRCULAR PLASMA CROSS- SECTIONS

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PLASMA CROSS-SECTIONS

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Abstract:

The axial contraction of linear pinches with racetrack-shaped and elliptic cross sections is studied by δW-analysis near the equilibrium shape. A surface current profile and a fixed plasma cross-section area are assumed. It is shown for the corresponding special perturbation that δW is given by the variation of the surface energy ($\delta W = \delta W_s = \frac{1}{8} B_p^2 \, \xi_b^2$), and that δW is nearly independent of the shape of the plasma (racetrack or ellipse). The axial oscillation frequency is found to be $\omega_{ax} = \frac{1}{2\sqrt{C}} \frac{B_p}{\sqrt{\rho_\ell}}$ with a constant $C < 1$. The model correctly predicts the B_p and ρ_ℓ – dependences of the experimental belt-pinch results and yields theoretical ω_{ax}-values a factor of 2 above the experimental ones.

1. INTRODUCTION

Axial contraction (parallel to the major torus axis) is observed in pinches with non-circular plasma cross sections, e.g. the belt-pinch. This occurs after the plasma has been produced in a non-equilibrium shape (with too large a plasma-wall distance) by shock heating and adiabatic compression, and it represents a transition to the equilibrium shape. This can cause a prolongation of the dynamic phase far beyond the time of crowbarring of the toroidal field B_t and of the poloidal field B_p. The experimental period with stationary conditions can be considerably reduced by this process.

In this paper, the axial oscillations around the equilibrium shape of linear pinches with racetrack-shaped and elliptic cross-sections are studied by δW - analysis.

2. δW -ANALYSIS OF THE AXIAL CONTRACTION FOR A RACETRACK-SHAPED CROSS-SECTION

Analysis in linear geometry is certainly sufficient because the toroidal curvature of the experimental plasma should not be important for these oscillations. A sharp pressure profile with surface currents parallel to the axis of the straight pinch is assumed ($B_p = 0$ and $B_t = 0$ in the plasma). The equilibrium is produced by image currents in parallel, perfectly conducting walls (see Fig.1) and is theoretically found for half-axis ratios above 2 at a plasma sheath thickness equal to one half of the wall distance /1/. At first a racetrack-shaped plasma cross section is assumed in agreement with the experiment (belt-pinch) with a half-axis ratio that varies during the axial oscillations (see Fig.1). At the equilibrium $d = a$ (a being the minor half-axis) and $h = b$ (b being the major half-axis) hold. In the neighbourhood of the equilibrium we define $d = a + \xi_a$ and $h = b + \xi_b$. For a sharp pressure profile the variation of the energy δW has three contributions: δW_p (plasma volume), δW_s (plasma surface) and δW_v (vacuum field region). It holds that

$$\delta W = \delta W_p + \delta W_s + \delta W_v \qquad (1)$$

According to /2/ the plasma contribution reads

$$\delta W_p = \frac{1}{2} \int \left\{ \frac{(\delta B)^2}{4\pi} + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \cdot (\vec{\xi} \times \delta\vec{B}) + (\vec{\nabla}\vec{\xi})\vec{\xi} \cdot \vec{\nabla} p + \gamma p (\vec{\nabla}\vec{\xi})^2 \right\} d\tau_p \qquad (2)$$

where $\delta\vec{B}$ is derived from Faraday's induction law and is given by

$$\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}) \qquad (3)$$

Here $\vec{\xi}$ is the perturbation vector, γ is the adiabatic exponent and $d\tau_p$ is a volume element in the plasma.

For the variation of the surface energy one obtains

$$\delta W_s = -\frac{1}{2} \int \xi_n^2 \left\langle \vec{\nabla} (p + \frac{B^2}{8\pi}) \right\rangle d\vec{S} \qquad (4)$$

$\vec{\xi}_n$ is the perturbation vector perpendicular to the surface element dS and is directed outwards. The expression $\langle A \rangle$ denotes the jump of the quantity A perpendicular to the surface and is defined as

$$\langle A \rangle = \lim_{\xi_n \to 0} \left[A(\vec{\xi}_n) - A(-\vec{\xi}_n) \right] \qquad (5)$$

For p, B and dS the undisturbed quantities must be inserted. δW_s represents the work that has to be done against the force due to the surface currents, when the interface between plasma and vacuum field region is displaced by $\vec{\xi}$.

Finally, the variation of the vacuum field energy reads

$$\delta W_v = \frac{1}{2} \int \frac{(\delta B)^2}{4\pi} d\tau_v \qquad (6)$$

where $d\tau_v$ is a volume element of the vacuum field region. In the experiment $B_t^2 \gg B_p^2$ holds, i.e. the plasma is confined by B_t. After crowbarring B_t is constant in time and in contrast to B_p does not vary with the axial oscillations. The consequence of a constant B_t is a fixed plasma cross-sectional area F_o. For large half-axis ratios ($\frac{b}{a} > 3$) the equilibrium shape of the plasma approximates a racetrack. According to Fig. 1 its cross-sectional area is given by

$$F_o = 4 d^2 (\frac{h}{d} + \frac{\pi}{4} - 1)$$

For $\frac{h}{d} \gg 0.2$ (e.g. $\frac{b}{a} \gtrsim 3$) F_o can be approximated very well by $F_o = 4 dh = 4 ab$.

From this it follows that in the neighbourhood of the equilibrium

$$\xi_a \simeq - \frac{a}{b} \xi_b (1 + \frac{\xi_b}{b})^{-1}$$

and for $\frac{\xi_b}{b} \ll 1$ one obtains

$$\xi_a \simeq -\frac{a}{b} \xi_b \tag{7}$$

Thus, the perturbation of the surface is defined for which the δW - analysis of a field-free plasma will be performed. If the interior of the plasma is not field-free, the perturbation function in the plasma must be specified in detail.

Substituting $\vec{B} = \vec{B}_t + \vec{B}_p$ in Eq. (3) yields a term $\vec{\nabla} \times (\vec{\xi} \times \vec{B}_t)$ that vanishes for the special perturbation defined above. Moreover, it can be shown that it also holds that $\vec{\nabla} \times (\vec{\xi} \times \vec{B}_p) = 0$. Consequently, $\delta \vec{B}$ and δW_v vanish in the vacuum field region. In the field-free plasma, too, $\delta \vec{B}$ is equal to zero.

For the special perturbation $\vec{\nabla} \vec{\xi} = 0$ holds and this yields $\delta W_p = 0$ since for a fixed F_o no compressional work is done against the plasma pressure. Therefore, in the second order considered here δW is determined by a variation of the surface energy δW_s alone.

The expression in pointed brackets in Eq. (4) represents a measure of the magnitude of the surface tension. Generally, the condition for MHD equilibrium reads

$$\vec{\nabla} p = \vec{j} \times \vec{B} = \frac{1}{4\pi} (\vec{B} \vec{\nabla}) \vec{B} - \vec{\nabla} \frac{B^2}{8\pi}$$

or

$$\vec{\nabla} (p + \frac{B^2}{8\pi}) = \frac{1}{4\pi} (\vec{B} \vec{\nabla}) \vec{B} \tag{8}$$

Thus, the surface tension is due to the $(\vec{B} \vec{\nabla}) \vec{B}$-force of the poloidal field that has a component normal to the plasma surface only in regions with curved field lines. This means for the racetrack-shape in particular that $(\vec{B} \vec{\nabla}) \vec{B}$-forces perpendicular to the plasma surface only exist at the semicircular ends. The poloidal field on the surface can be approximated at the ends by the field on a cylinder, especially, because in equilibrium the current density on the racetrack is constant like that on a cylinder. For the normal component of this force one thus obtains

$$< \nabla_n (p + \frac{B^2}{8\pi}) > = - \frac{1}{4\pi} \frac{<B_p^2>}{a}$$

According to Eq. (5) one finds $< B_p^2 > = B_p^2$,

where B_p is the value on the surface that is constant for the equilibrium shape. This yields

$$\delta W_s = \frac{1}{8\pi} \frac{B_p^2}{a} \int \xi_n^2 \, dS$$

The normal component of $\vec{\xi}$ at the ends of the racetrack is calculated by means of Fig. 1. For the half-axis ratios that lead to a racetrack-shape $\xi_a \ll \xi_b$ holds so that $\vec{\xi}$ is essentially an outward displacement of the ends by the distance ξ_b. According to Fig. 1 one obtains $\varepsilon_n = \varepsilon_b \sin \alpha$ and the integral

$$\int_{\alpha = 0}^{2\pi} \xi_b^2 \sin^2 \alpha \, a d\alpha = a \, \xi_b^2 \, \pi$$

This finally yields

$$\delta W = \delta W_s = \frac{1}{8} B_p^2 \, \xi_b^2 \tag{9}$$

This expression, in fact, exceeds δW_v by one order of magnitude. This can be seen from a detailed computation of $\delta W_v / 1 /$ that yields

$$\delta W_v \approx \frac{1}{8\pi} B_p^2 \frac{a}{b} \, \xi_b^2$$

It is concluded that the axial oscillations around the equilibrium shape are caused by a variation of the energy at the plasma surface.

For a conservative system, i.e. a system with no dissipation of energy and no exchange of energy with the outer electrical circuit, one obtains

$$\frac{1}{2} \int \rho \left(\frac{d\xi}{dt} \right)^2 d\tau_p + \delta W_s = 0 \tag{10}$$

This equation can be used to determine the frequency of the axial oscillations (ω_{ax}) by replacing the kinetic energy by

$$\frac{1}{2} \int \rho \left(\frac{d\xi}{dt} \right)^2 d\tau_p = \frac{1}{2} C\rho_\ell \left(\frac{d\xi_b}{dt} \right)^2 \tag{11}$$

where C is a dimensionless constant factor smaller than unity, and ρ_ℓ is the line mass. The real motion of the plasma is replaced by the oscillation $\xi_b (\omega_{ax} t)$ of a mass $C\rho_\ell$ that is concentrated at the ends of the racetrack. For a harmonic oscillation

$$\xi_b = (\xi_b)_o \, e^{i \, \omega_{ax} t}$$

one finds by substituting Eqs.(9) and (11) in Eq. (10)

$$-\frac{1}{2} C\rho_\ell \ ^{(1)}\omega_{ax}^2 \xi_b^2 + \frac{1}{8} B_p^2 \xi_b^2 = 0$$

$$^{(1)}\omega_{ax} = \frac{1}{2\sqrt{C}} \frac{B_p}{\sqrt{\rho_\ell}} \tag{12}$$

3. δW – ANALYSIS OF THE AXIAL CONTRACTION FOR AN ELLIPTIC CROSS SECTION

This section deals with the question whether the variation of energy sensitively depends on the shape of the cross-section or not. For this purpose δW_s is computed for an elliptic cross-section (see Fig.2). The calculation is performed in elliptic coordinates:

$$x = c \cos h \ \Psi \ \cos \ \chi$$
$$y = c \sin h \ \Psi \ \sin \ \chi$$
$$z = z$$

For the Ψ , χ and z-components we set the indices 1, 2 and 3 respectively. For elliptic coordinates it holds that $h_1^2 = h_2^2 = c^2 (\sin h^2 \Psi + \sin^2 \chi)$ and $h_3^2 = 1$. Moreover, we set $B_2 = B_p$ and $B_3 = B_t$, i.e. $B^2 = B_2^2 + B_3^2$. For the equilibrium flux surface $\Psi = \Psi_o$ in particular one obtains

$$x = c \cos h \ \Psi_o \ \cos \chi = b \cos \ \chi$$
$$y = c \sin h \ \Psi_o \ \sin \ \chi = a \sin \ \chi \tag{13}$$
$$z = z$$

The computation of $\vec{j} \times \vec{B}$ in elliptic coordinates yields an identity that corresponds to Eq. (8):

$$\frac{1}{h_1} \frac{\partial}{\partial \Psi} (p + \frac{B^2}{8\pi}) = - \frac{1}{4\pi} \frac{B_2^2}{h_1^2} \frac{\partial h_1}{\partial \Psi}$$

Since it holds that $\dfrac{\partial h_1}{\partial \Psi} = \dfrac{c^2}{2} \dfrac{\sin h(2\Psi)}{h_1}$, one obtains

$$\nabla_n (p + \frac{B^2}{8\pi}) = - \frac{c^2}{8\pi} B_2^2 \frac{\sin h (2\Psi)}{h_1^3} \tag{14}$$

A perturbation of the ellipse $\Psi = \Psi_o$ is now assumed that is analogous to the m = 2 perturbation of a circle and that reads

$$\xi = r(\chi)\, \epsilon \,\cos(2\chi)$$

and

$$r(\chi) = \sqrt{x^2 + y^2} = \sqrt{b^2 \cos^2\chi + a^2 \sin^2\chi}$$

i.e. ξ is proportional to the length of the radius vector. According to Eq. (13) $\xi = b \cdot \epsilon = \xi_b$ for $\chi = 0$ and $\xi = -a\,\epsilon = \xi_a$ for $\chi = \frac{\pi}{2}$. This yields $\epsilon = \frac{\xi_b}{b} = -\frac{\xi_a}{a}$, i.e. the condition for a constant cross-sectional area of the ellipses for $\frac{\xi_b}{b} \ll 1$. This is identical with Eq. (7). For $\chi = \frac{\pi}{4}$ one finds $\xi = 0$, and this marks the points of intersection of the ellipse $\psi = \psi_o$ with the perturbed ellipse. According to Eq. (4) the scalar product of the gradient and $\vec{dS}$ is given by

$$-\langle \nabla_n (p + \frac{B^2}{8\pi})\rangle \; dS \text{ in the range } 0 \le \chi < \frac{\pi}{4} \text{ and}$$

$$\langle \nabla_n (p + \frac{B^2}{8\pi})\rangle \; dS \text{ in the range } \frac{\pi}{4} < \chi \le \frac{\pi}{2} . \text{ From Eqs. (5) and (14) one}$$

obtains

$$\langle \nabla_n (p + \frac{B^2}{8\pi})\rangle = -\frac{c^2}{8\pi} B_p^2 \frac{\sin h(2\psi_o)}{h_1^3(\psi_o)}$$

Substituting this in Eq. (4) yields with $dS = h_1(\psi_o)\, d\chi$

$$\delta W_s = \frac{1}{2} \int_{\text{surface}} \xi_n^2 \frac{c^2}{8\pi} B_p^2 \frac{\sin h(2\psi_o)}{h_1^2(\psi_o)} d\chi$$

The following identities are easily proven:

$$c^2 \sin h(2\psi_o) = 2ab$$

$$h_1(\psi_o) = a\sqrt{\cos^2\chi + (\frac{b}{a})^2 \sin^2\chi}$$

$$\xi = \xi_b \sqrt{\cos^2\chi + (\frac{a}{b})^2 \sin^2\chi} \;\; \cos(2\chi)$$

With $\xi_n = \xi \sin\alpha$ (see Fig. 2) one finds

$$\delta W_s = \frac{1}{8\pi} ab\, B_p^2 \xi_b^2 \int_{\text{surface}} \frac{[\cos^2\chi + (\frac{a}{b})^2 \sin^2\chi]\,\cos^2(2\chi)\,\sin^2\alpha\, d\chi}{a^2 [\cos^2\chi + (\frac{b}{a})^2 \sin^2\chi]}$$

The angle α at the point $P(x,y)$ is given by

$$\alpha = \alpha_1 + \alpha_2 \quad \text{and}$$

$$\alpha_1 = \tan^{-1}\left(\frac{a}{b}\tan\chi\right) \quad \text{and}$$

$$\alpha_2 = \tan^{-1}\left(\frac{a}{b}\cot\chi\right)$$

This yields

$$\sin^2\alpha = \frac{A^2}{A^2 + \sin^2(2\chi)} \quad \text{and}$$

$$A = \frac{2\frac{a}{b}}{1 - \left(\frac{a}{b}\right)^2} \tag{15}$$

Finally, one obtains

$$\delta W_s = \frac{1}{8} B_p^2 \, \xi_b^2 \cdot F\left(\frac{b}{a}\right) \tag{16}$$

and

$$F\left(\frac{b}{a}\right) = \frac{4}{\pi} \frac{b}{a} \left[\int_{\chi=0}^{\frac{\pi}{4}} G(\chi)\, d\chi - \int_{\chi=\frac{\pi}{4}}^{\frac{\pi}{2}} G(\chi)\, d\chi \right]$$

where $G(\chi)$ is given by

$$G(\chi) = \frac{\cos^2\chi + \left(\frac{a}{b}\right)^2 \sin^2\chi}{\cos^2\chi + \left(\frac{b}{a}\right)^2 \sin^2\chi} \cdot \frac{A^2 \cos^2(2\chi)}{A^2 + \sin^2(2\chi)}$$

Apart from the factor $F\left(\frac{b}{a}\right)$ this result is identical with Eq. (9). Numerical integration yields the curve in Fig. 3. It is seen that $F\left(\frac{b}{a}\right)$ is about 0.8 for $\frac{b}{a} = 3$ and 0.9 for $\frac{b}{a} = 5$. Obviously δW_s is nearly independent of the shape of the plasma (elliptic or racetrack-like cross-section), and Eq. (12) is valid.

4. COMPARISON WITH EXPERIMENTAL RESULTS

The theoretically derived frequency shall now be compared with the experiment (belt-pinch I) that exhibits this axial contraction with a fixed plasma cross-sectional area. On the basis of smear pictures of this motion the dependence of ω_{ax} on B_p is studied for a filling pressure p_o = 35 mtorr (D_2) and for a maximum toroidal field B_t = 6.5 kG. The result is shown in Fig.4. It is found that ω_{ax} scales in proportion with B_p at a fixed line mass. The variation of ω_{ax} with ρ_ℓ also agrees well with the theoretical dependence in Eq. (12). If C is set equal to 0.7 in Eq. (12),

one obtains absolute ω_{ax}-values that exceed the experimental results just by a factor of 2. The agreement is sufficient if one takes into account that the experiment differs from the idealized assumptions of the model calculation in several respects. Thus, for example, the experimental plasma profile is diffuse, the plasma is not field-free, and the axial oscillation is strongly damped. Moreover, the experimental motion of the plasma ends is linear in time rather than harmonic. The reason for this is that the mass at the ends is picked up during contraction, and this corresponds to a time-dependent quantity C.

It shall be noted here that a compressional wave due to B_t in the direction of the large half-axis should roughly have a time scale $\tau \approx \dfrac{b}{v_A}$ (v_A Alfvén velocity), which is found experimentally. However, the experimental dependence of ω_{ax} on B_p excludes this process.

5. SUMMARY

The axial oscillations around the equilibrium shape of a linear pinch (analogous to an m = 2 mode) have been studied by a δW - analysis for a racetrack-shaped and for an elliptic cross-section. A surface current profile and a fixed plasma cross-sectional area are assumed. It has been shown that δW is given by δW_s, i.e. by the variation of the surface energy ($\delta W = \delta W_s = \dfrac{1}{8} B_p^2 \xi_b^2$). Moreover, it has been demonstrated that δW is nearly independent of the shape of the plasma (racetrack or ellipse). For the axial oscillation frequency $\omega_{ax} = \dfrac{1}{2\sqrt{C}} \dfrac{B_p}{\sqrt{\rho_\ell}}$ with a constant C< 1 is found.
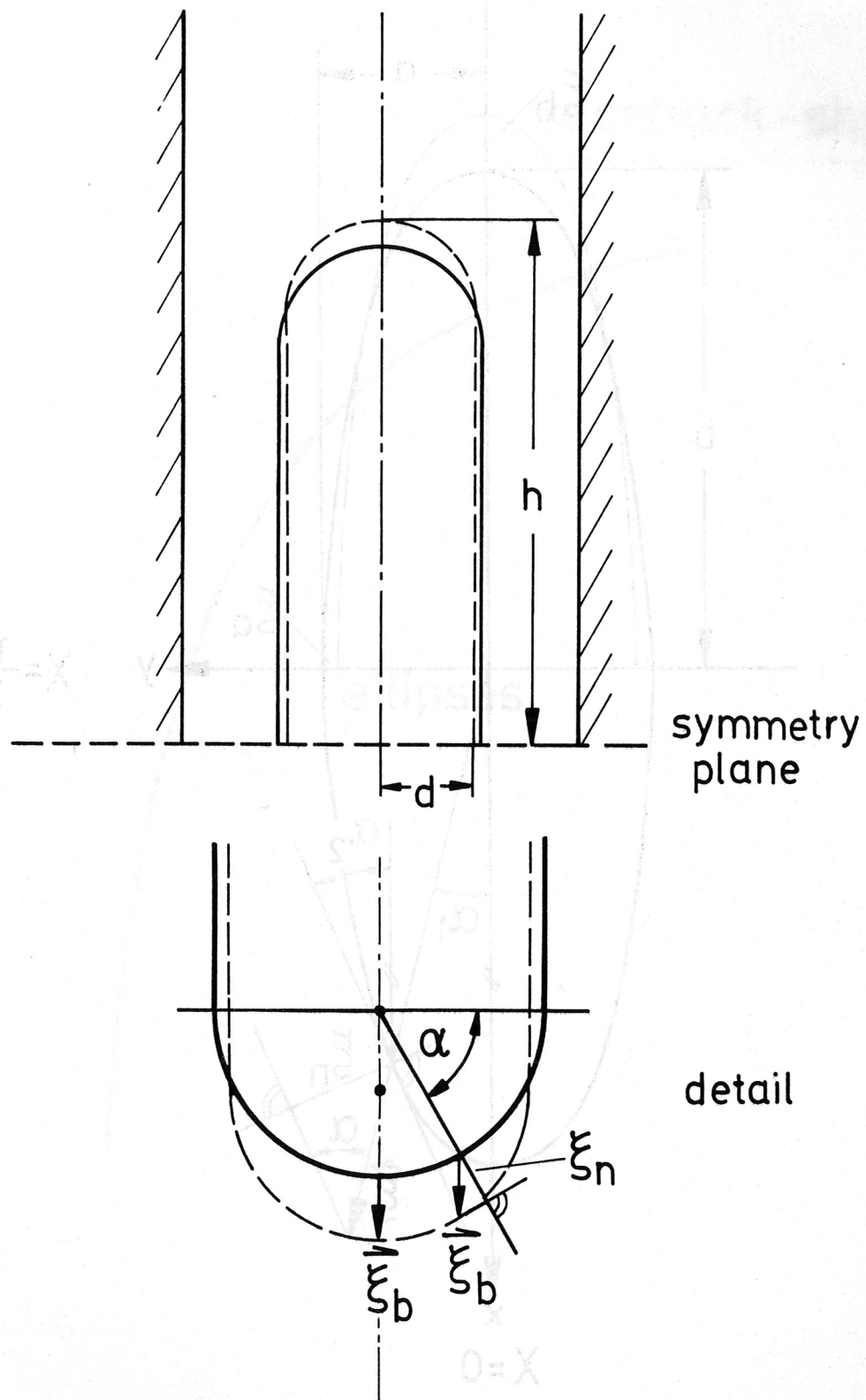
A comparison with experimental belt-pinch results has shown that the B_p and ρ_ℓ - dependences of ω_{ax} are predicted correctly by the model, and that the theoretical ω_{ax} -values exceed the experimental ones by a factor of 2.

Acknowledgements

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Figure captions:

Fig. 1   Racetrack-shaped equilibrium plasma cross-section and perturbed surface.

Fig. 2   Elliptic equilibrium plasma cross-section and perturbed surface.

Fig. 3   Factor F in δW_s versus $\frac{b}{a}$ for elliptic cross-sections.

Fig. 4   Experimental dependence of ω_{ax} on B_p for a filling pressure $p_o = 35$ mtorr(D_2) and a maximum toroidal field $B_t = 6.5$ kG.

symmetry
plane

detail

FIG. 1

FIG. 2

FIG. 3

FIG. 4